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ADDENDUM

Icosians versus octonions as descriptions of the E_8 lattice

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Abstract. A simple method for the construction of the E_8 root system with icosians is suggested. It is confronted with the root system of E_8 obtained by integral octonions.

Symmetries of the root lattice of E_8 and its sublattices play important roles in the compactification [1] of the heterotic string [2]. It is therefore desirable to investigate its properties by all relevant mathematical means. In a companion work [3] we have constructed the E_8 lattice with integral octonions [4] and displayed its branching with respect to its maximal subalgebras.

In this addendum to [3] we construct the root system of E_8 with icosians [5] using a similar method as in [3] and determine the relations between these two types of construction. The icosian is the generic name for the set of 120 elements of the binary icosahedral group, a double cover of the icosahedral group (isomorphic to the group A_5 of even permutations of five letters), consisting of the quaternions [5]:

$$\pm 1, \pm e_1, \pm e_2, \pm e_3 \qquad \tfrac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) \tag{1a}$$

$$\begin{array}{lll} \tfrac{1}{2}(\pm\tau \pm e_1 \pm \sigma e_2) & \tfrac{1}{2}(\pm\tau \pm e_2 \pm \sigma e_3) & \tfrac{1}{2}(\pm\tau \pm \sigma e_1 \pm e_3) \\ \tfrac{1}{2}(\pm 1 \pm \tau e_1 \pm \sigma e_3) & \tfrac{1}{2}(\pm 1 \pm \sigma e_1 \pm \tau e_2) & \tfrac{1}{2}(\pm 1 \pm \sigma e_2 \pm \tau e_3) \\ \tfrac{1}{2}(\pm\sigma \pm \tau e_2 \pm e_3) & \tfrac{1}{2}(\pm\sigma \pm e_1 \pm \tau e_3) & \tfrac{1}{2}(\pm\sigma \pm \tau e_1 \pm e_2) \\ \tfrac{1}{2}(\pm\sigma e_1 \pm e_2 \pm \tau e_3) & \tfrac{1}{2}(\pm\tau e_1 \pm \sigma e_2 \pm e_3) & \tfrac{1}{2}(\pm e_1 \pm \tau e_2 \pm \sigma e_3) \end{array} \tag{1b}$$

where τ and σ are defined by

$$\begin{array}{llll} \tau = \tfrac{1}{2}(1+5) & \sigma = \tfrac{1}{2}(1-5) & & \\ \tau + \sigma = 1 & \tau^2 = \tau + 1 & \sigma^2 = \sigma + 1 & \tau\sigma = -1 \end{array} \tag{2}$$

and e_1, e_2 and e_3 are the quaternionic imaginary units satisfying

$$e_i e_j = -\delta_{ij} + \varepsilon_{ijk} e_k \qquad (i, j, k = 1, 2, 3) \qquad \bar{e}_i = -e_i \tag{3}$$

where δ_{ij} and ε_{ijk} are the usual Kronecker and Levi-Civita symbols respectively. 24 of the integral quaternions in (1a) (Hurwitz integers) [6] form the binary tetrahedral group and represent the root system of SO(8). 120 of the elements in (1a) and (1b) can be generated by two elements $A = \tfrac{1}{2}(\tau - \sigma e_1 + e_3)$ and $B = \tfrac{1}{2}(1 - \sigma e_2 + \tau e_3)$ satisfying the relations

$$A^5 = B^3 = C^2 = ABC = -1 \tag{4}$$

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with $C = e_3$. Let p and q be two quaternions. We define the scalar product by the relation

$$(p, q) = \tfrac{1}{2}(\bar{p}q + \bar{q}p). \tag{5}$$

With this definition, the scalar product of icosians in ($1a$) and ($1b$) will take the values $a + b\sigma$ where a and b are 0, $\pm\frac{1}{2}$, ± 1. Wilson and Conway in [5] define a 'reduced' scalar product by the mapping $a + b\sigma \to a$. With this new definition of the scalar product they show that the set of icosians q and σq form the root system of E_8. Moroever, with the same definition of 'reduced' scalar product they construct the Leech lattice [7] with icosians.

Following [3] we denote the roots and three eight-dimensional representations of SO(8) by the sets of integral and half integral quaternions

$$\begin{array}{cccc} A_0 & A_1 & A_2 & A_3 \\ 24{:}\pm 1, \pm e_1, \pm e_2, \pm e_3 & 8_v{:}\frac{1}{2}(\pm 1 \pm e_1) & 8_c{:}\,\frac{1}{2}(\pm 1 \pm e_2) & 8_s{:}\,\frac{1}{2}(\pm 1 \pm e_3) \\ \frac{1}{2}(\pm 1 \pm e_1 \pm e_2 \pm e_3) & \frac{1}{2}(\pm e_2 \pm e_3) & \frac{1}{2}(\pm e_3 \pm e_1) & \frac{1}{2}(\pm e_1 \pm e_2). \end{array} \tag{6}$$

These are the 48 roots of F_4. Here A_1, A_2 and A_3 denote the short roots, A_0 represents the long roots of F_4. In [8] it has been shown that by 'matching' two system of F_4 roots one can construct the E_8 lattice. This method is used in [3] to express the E_8 lattice in terms of octonions. Here we use the same trick. To obtain 240 roots of E_8 we multiply the F_4 roots in (6) by σ and add them to the roots in (6) provided the so-obtained icosians have the 'reduced' norm of unity. In fact the following set of icosians

$$\begin{array}{lll} (A_0, 0) \equiv A_0 & (0, A_0) \equiv \sigma A_0 & (A_1, A_2) = A_1 + \sigma A_2 \\ (A_2, A_3) = A_2 + \sigma A_3 & (A_3, A_1) = A_3 + \sigma A_1 & \end{array} \tag{7}$$

not only reproduces the 120 elements in ($1a$) and ($1b$), denoted by q, but also yields the additional 120 roots σq of E_8. It is clear from the definition of 'reduced' scalar product that the quaternionic units 1, e_1, e_2, e_3 and σ, σe_1, σe_2, σe_3 form an orthogonal set of eight elements. A Coxeter-Dynkin diagram leading to the E_8 roots in (7) are given in figure 1. In [3] we have given the octonionic roots of E_8 with the notation

$$\begin{array}{lll} [A_0, 0] \equiv A_0 & [0, A_0] \equiv e_7 A_0 & [A_1, A_1] \equiv A_1 + e_7 A_1 \\ [A_2, A_3] \equiv A_2 + e_7 A_3 & [A_3, A_2] \equiv A_3 + e_7 A_2. & \end{array} \tag{8}$$

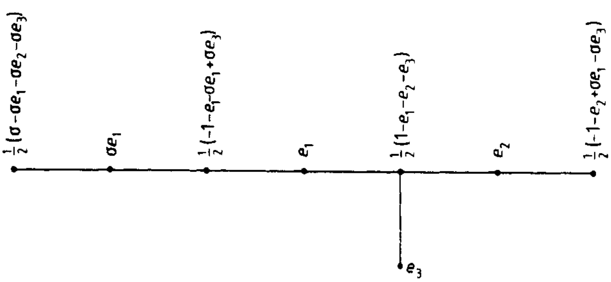

Figure 1. Coexter-Dynkin diagram of E_8 with icosians.

A comparison of (7) and (8) suggests that a mapping between the octonionic roots and the icosians can be obtained in the following form:

$$\begin{array}{lcl} \sigma A_2 \leftrightarrow e_7 A_1 & & \sigma \leftrightarrow e_7 \\ \sigma A_1 \leftrightarrow e_7 A_2 & \rightarrow & \sigma e_3 \leftrightarrow e_6 = e_7 e_3 \\ \sigma A_3 \leftrightarrow e_7 A_3 & & \sigma e_1 \leftrightarrow e_5 = e_7 e_2 \\ & & \sigma e_2 \mapsto e_4 = e_7 e_1 . \end{array} \tag{9}$$

With the obvious mapping $1 \leftrightarrow 1$, $e_1 \leftrightarrow e_1$, $e_2 \leftrightarrow e_2$, $e_3 \leftrightarrow e_3$ one can easily transform one construction to the other. The correspondence in (9) will also lead to the octonionic construction of the Leech lattice described by Wilson in (5). In appendix 1 of [3] we have shown that two more independent octonionic root systems of E_8 can be obtained by successive applications of the replacement $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_1$ in (8). These changes will certainly alter the mappings in (9) accordingly.

The algebraic structures of the root systems of E_8 with octonions and icosians can be confronted as follows.

(i) The octonionic root system obeys the usual scalar product defined by (5) and forms a closed non-associative algebra of 240 elements, only 24 of which form a group called the binary tetrahedral group.

(ii) 120 elements of icosians in (7) form the binary icosahedral group extending the order of the group structure in the case of octonions, but the whole set of 240 icosians do not close under multiplication since $(\sigma q)(\sigma q') = q + \sigma q'$ produces the lattice vectors of higher norm.

(iii) As we have shown in [9] octonionic roots of E_8 yield natural Abelian symmetries Z_6, Z_4, Z_3 and Z_2 of the E_8 lattice and an interesting manifestation of the triality of the extended Coxeter-Dynkin diagram of E_6.

With icosianic roots, while preserving the triality structure of the extended Coxeter-Dynkin diagram of E_6 one can naturally have the Abelian symmetries Z_{10}, Z_6, Z_5, Z_4, Z_3, Z_2 of the root system of E_8. To be more specific the maximal subgroup $SU(5) \times SU(5)$ of E_8 can be embedded in E_8 with a Z_5 symmetry invariance a case, which is not possible in the octonionic representation of E_8 lattice.

Another amusing observation is the possibility of describing the $E_8 \times E_8'$ root system by a simple extension of the root system in (7). Indeed, if we multiply the icosians in (7) by e_7 we obtain a root system of 240 elements described by the octonionic units e_4, e_5, e_6, e_7 and σe_4, σe_5, σe_6 and σe_7. This second set of 240 octonionic elements can be used to describe the root lattice of an independent E_8'. Its algebraic structure is also interesting. The product of any two roots from E_8' will give a root or a vector of higher norm in E_8. A more important aspect is that the roots of E_8' are non-associative under multiplication. This feature of E_8' could be attributed to the reason why E_8' is not broken in the heterotic string [2] by compactification. The reason could be algebraic rather than dynamical.

Details of this work, emphasising more on the Z_5 symmetry of the icosians will be published elsewhere [10].

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